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Computational Method for the direct Solution of Second Order Oscillatory Betiss and Stiefel Differential Equation

Adamu A. Bambur[1*], Nondika Paul[2] & Raliatu Kashim[3]

[1,2]Department of Mathematics and Statistics, Taraba State Polytechnic Suntai, Jalingo Campus, Taraba State, Nigeria.
[3]Department of Mathematics and Statistics, Abubakar Tatari Ali Polytechnic, Bauchi, Nigeria.
Corresponding Author (Adamu A. Bambur) Email: adamsbambur@gmail.com[*]

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ABSTRACT

This study presents an optimized computational method for the direct solution of second-order oscillatory differential equations, specifically the Betiss and Stiefel forms. The method uses power series-based interpolation together with collocation techniques to create a continuous linear multistep hybrid modeling system. The theoretical analysis proves that the method achieves high order accuracy together with all required mathematical properties, which include consistency and zero-stability and convergence and a specified domain of absolute stability. The proposed method demonstrates high accuracy when tested against exact solutions and existing numerical methods, which both graphical and tabular results confirm as reliable and efficient. The method demonstrates itself as a strong and efficient solution method for second-order nonlinear differential equations, which appear in applied mathematics and physics and engineering fields.

Keywords: Second-Order; Oscillatory Differential Equations; Betiss Differential Equation; Stiefel Differential Equation; Linear Multistep Method; Numerical Solution; Convergence; Stability Analysis; Power Series; Collocation Method; Interpolation Method.

1. Introduction

The field of mathematical modeling has developed into a essential instrument that scientists use today to explore biological and physical and medical processes especially in their research on dynamic systems and body cooling and simple harmonic motion. The mathematical models which scientists developed have led to progress in mathematical theory and bioscience research according to author [1, 2]. The application of mathematics in these fields has fostered innovative approaches and opened new opportunities for interdisciplinary collaboration. The mathematical methods deliver their best results during dynamic problem solving and thermal process research because they produce better understanding and accurate future outcomes [3]. The study aims to solve initial value problems (IVPs) for second-order ordinary differential equations (ODEs) using the following equation format:

$$\frac{d^2y}{dt^2} = f(t, y, y'),, y(0) = \lambda_1, \frac{dy}{dt} = \lambda_2, t \in [t_0, t_n] \qquad (1)$$

where t_0 represents initial value/point, y_0 denotes the solution at time t_0, f remains continuous over the integration interval. We operate under the assumption that equation (1) adheres to the existence and uniqueness theorem of differential equations. Furthermore, we presume that solutions to equations akin to (1) remain bounded. It's crucial to clarify that a solution $y(t)$ to equation (1.1) is deemed bounded if,

$$\sup_{t \in \Re} \| y(t) \| < \infty \qquad (2)$$

The numerical treatment of equations like (1) has remained a strong research focus [4-7].

The research demonstrated that direct solution methods for equation (1) provide better efficiency than solving it through conversion to first-order ordinary differential equation systems according to the findings presented in reference [8]. The academic community has responded to this discovery by investigating direct resolution techniques which enable them to solve equation (1) without applying reduction methods. The literature contains multiple methods which researchers have developed to solve equation (1) through direct approaches. The works of Ref. [9-12] represent important contributions to the research. The research of [13-15] represents another major investigation of oscillating differential equations. Researchers such as [16-18] have worked on solving second order initial value problems through methods which avoid first order initial value problem reductions. The studies demonstrate an increasing demand for direct methods which can solve second-order ordinary differential equations with efficient results especially for the Betiss and Stiefel Linear Oscillatory Differential Equation represented by equation (1).

1.1. Study Objectives

The following are objectives of the study:

1) To derive a computational method using power series polynomial.
2) To continuous hybrid linear multistep method.
3) To study the analysis of the basic properties of the computational method.
4) To test the efficiency of the method on some second order oscillatory differential equations.
5) To shows the comparison in a table and graphical shown.

2. Mathematical Formulation of Computational Method

An approximate solution to a power series polynomial of the for

$$y(t)=\sum_{j=0}^{p+q-1}\alpha_j t^j \qquad (3)$$

is considered as a basis function for the direct solution of the second initial value problems of the form (1). Where $t\in[a,b]$, the $a's$ are real unknown parameters to be determined and $p+q$ is the sum of the number of interpolation and collocation points.

differentiating (3) twice, yield

$$y''(t)=\sum_{j=0}^{u+v-1}j(j-1)a_j t^{j-2} \qquad (4)$$

Now, interpolating (3) at point $p=\frac{1}{2},1$ and collocating (4) at $q=0,\frac{1}{6},\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{3}{4},1$ which lead to a system of equation in a matrix form as

$$\begin{bmatrix}
1 & t_{n+\frac{1}{2}} & t^2_{n+\frac{1}{2}} & t^3_{n+\frac{1}{2}} & t^4_{n+\frac{1}{2}} & t^5_{n+\frac{1}{2}} & t^6_{n+\frac{1}{2}} & t^7_{n+\frac{1}{2}} & t^8_{n+\frac{1}{2}} & t^9_{n+\frac{1}{2}} \\
1 & t_{n+1} & t^2_{n+1} & t^3_{n+1} & t^4_{n+1} & t^5_{n+1} & t^6_{n+1} & t^7_{n+\frac{1}{2}} & t^8_{n+1} & t^9_{n+1} \\
0 & 0 & 2 & 6t_n & 12t^2_n & 20t^3_n & 30t^4_n & 42t^5_n & 56t^6_n & 72t^7_n \\
0 & 0 & 2 & 6t_{n+\frac{1}{6}} & 12t^2_{n+\frac{1}{6}} & 20t^3_{n+\frac{1}{6}} & 30t^4_{n+\frac{1}{6}} & 42t^5_{n+\frac{1}{6}} & 56t^6_{n+\frac{1}{6}} & 72t^7_{n+\frac{1}{6}} \\
0 & 0 & 2 & 6t_{n+\frac{1}{4}} & 12t^2_{n+\frac{1}{4}} & 20t^3_{n+\frac{1}{4}} & 30t^4_{n+\frac{1}{4}} & 42t^5_{n+\frac{1}{4}} & 56t^6_{n+\frac{1}{4}} & 72t^7_{n+\frac{1}{4}} \\
0 & 0 & 2 & 6t_{n+\frac{1}{3}} & 12t^2_{n+\frac{1}{3}} & 20t^3_{n+\frac{1}{3}} & 30t^4_{n+\frac{1}{3}} & 42t^5_{n+\frac{1}{3}} & 56t^6_{n+\frac{1}{3}} & 72t^7_{n+\frac{1}{3}} \\
0 & 0 & 2 & 6t_{n+\frac{1}{2}} & 12t^2_{n+\frac{1}{2}} & 20t^3_{n+\frac{1}{2}} & 30t^4_{n+\frac{1}{2}} & 42t^5_{n+\frac{1}{2}} & 56t^6_{n+\frac{1}{2}} & 72t^7_{n+\frac{1}{2}} \\
0 & 0 & 2 & 6t_{n+\frac{2}{3}} & 12t^2_{n+\frac{2}{3}} & 20t^3_{n+\frac{2}{3}} & 30t^4_{n+\frac{2}{3}} & 42t^5_{n+\frac{2}{3}} & 56t^6_{n+\frac{2}{3}} & 72t^7_{n+\frac{2}{3}} \\
0 & 0 & 2 & 6t_{n+\frac{3}{4}} & 12t^2_{n+\frac{3}{4}} & 20t^3_{n+\frac{3}{4}} & 30t^4_{n+\frac{3}{4}} & 42t^5_{n+\frac{3}{4}} & 56t^6_{n+\frac{3}{4}} & 72t^7_{n+\frac{3}{4}} \\
0 & 0 & 2 & 6t_{n+1} & 12t^2_{n+1} & 20t^3_{n+1} & 30t^4_{n+1} & 42t^5_{n+1} & 56t^6_{n+1} & 72t^7_{n+1}
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix}
=
\begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+2} \\ f_n \\ f_{n+\frac{1}{6}} \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{2}{3}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{bmatrix} \qquad (5)$$

using Gaussian elimination method, (5) is solved for the a_j's. The values of the a_j's obtained are then substituted into (1), after some manipulations, this gives a continuous hybrid linear multistep method of the form;

$$y(t) = \alpha_{\frac{1}{2}}(t) y_{n+\frac{1}{2}} + \alpha_1(t) y_{n+1} + h^2 \left[\sum_{j=0}^{1} \beta_j(t) f_{n+j} + \beta_{v_i}(t) f_{n+v_i} \right], v_i = 0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \qquad (6)$$

the coefficient $\alpha_{\frac{1}{2}}, \alpha_1, \beta_0, \beta_{\frac{1}{6}}, \beta_{\frac{1}{4}}, \beta_{\frac{1}{3}}, \beta_{\frac{1}{2}}, \beta_{\frac{2}{3}}, \beta_{\frac{3}{4}}, \beta_1$ are given by;

$$\left.\begin{aligned}
\alpha_{\frac{1}{2}} &= 2 - 2t \\
\alpha_1 &= 2t - 1 \\
\beta_0 &= \frac{41}{13440} - \frac{1867}{40320} t + \frac{1}{2} t^2 - \frac{113}{36} t^3 + \frac{427}{36} t^4 - \frac{3367}{120} t^5 + \frac{1876}{45} t^6 - \frac{113}{3} t^7 + \frac{132}{7} t^8 - 4t^9 \\
\beta_{\frac{1}{6}} &= -\frac{54}{175} t + \frac{648}{35} t^3 - \frac{594}{5} t^4 + \frac{9072}{25} - \frac{3132}{5}\ t^6 + \frac{21816}{35} t^7 - \frac{11664}{35} t^8 + \frac{2592}{35} t^9 \\
\beta_{\frac{1}{4}} &= \frac{2}{21} + \frac{26}{175} t - \frac{512}{15} t^3 + \frac{11392}{45} t^4 - \frac{21248}{25} t^5 + \frac{14080}{9} t^6 - \frac{171008}{105} t^7 + \frac{31488}{35} t^8 - \frac{1024}{5} t^9 \\
\beta_{\frac{1}{3}} &= -\frac{243}{4480} - \frac{567}{3200} t + \frac{243}{10} t^3 - \frac{1539}{8} t^4 + \frac{138267}{200} t^5 - \frac{6723}{5} t^6 + \frac{51111}{35} t^7 - \frac{5832}{7} t^8 + \frac{972}{5} t^9 \\
\beta_{\frac{1}{2}} &= \frac{17}{105} - \frac{9}{35} t - 8t^3 + \frac{202}{3} t^4 - \frac{1304}{5} t^5 + \frac{8252}{15} t^6 - \frac{4504}{7} t^7 + \frac{2736}{7} t^8 - 96t^9 \\
\beta_{\frac{2}{3}} &= -\frac{243}{4480} + \frac{1647}{22400} t + \frac{81}{20} t^3 - \frac{351}{10} t^4 + \frac{28269}{200} t^5 - \frac{1566}{5} t^6 + \frac{13581}{35} t^7 - \frac{8748}{35} t^8 + \frac{324}{5} t^9 \\
\beta_{\frac{3}{4}} &= \frac{2}{21} - \frac{278}{1575} t + \frac{512}{315} t^3 - \frac{128}{9} t^4 + \frac{4352}{75} t^5 - \frac{5888}{45} t^6 + \frac{17408}{105} t^7 - \frac{768}{7} t^8 - \frac{1024}{35} t^9 \\
\beta_1 &= \frac{41}{13440} - \frac{143}{22400} t + \frac{1}{30} t^3 - \frac{107}{360} t^4 + \frac{249}{200} t^5 - \frac{131}{45} t^6 + \frac{407}{405} t^7 - \frac{96}{35} t^8 + \frac{4}{5} t^9
\end{aligned}\right\} \qquad (7)$$

evaluating (6) at non interpolating points to obtain the continuous form as,

$$\left.\begin{aligned}
&y_n-2y_{n+\frac{1}{2}}-y_{n+1}=\frac{41}{13440}h^2f_n+\frac{2}{21}h^2f_{n+\frac{1}{4}}-\frac{243}{4480}h^2f_{n+\frac{1}{3}}+\frac{17}{105}h^2f_{n+\frac{1}{2}}-\frac{243}{4480}h^2f_{n+\frac{2}{3}}+\frac{2}{21}f_{n+\frac{3}{4}}+\frac{41}{13440}f_{n+1}\\
&y_{n+\frac{1}{6}}-\frac{5}{3}y_{n+\frac{1}{2}}-+\frac{2}{3}y_{n+1}=\frac{8851}{8817984}h^2f_n-\frac{188}{8505}h^2f_{n+\frac{1}{6}}+\frac{52522}{688905}h^2f_{n+\frac{1}{4}}-\frac{4979}{90720}h^2f_{n+\frac{1}{3}}+\frac{10117}{91854}h^2f_{n+\frac{1}{2}}-\frac{20473}{544320}h^2f_{n+\frac{2}{3}}+\frac{44146}{688905}h^2f_{n+\frac{3}{4}}+\frac{22283}{11022480}h^2f_{n+1}\\
&y_{n+\frac{1}{4}}-\frac{3}{2}y_{n+\frac{1}{2}}+\frac{1}{2}y_{n+1}=\frac{1749}{2293760}h^2f_n-\frac{4941}{286720}h^2f_{n+\frac{1}{6}}+\frac{359}{6720}h^2f_{n+\frac{1}{4}}-\frac{106191}{2293760}h^2f_{n+\frac{1}{3}}+\frac{23393}{286720}h^2f_{n+\frac{1}{2}}-\frac{64233}{2293760}h^2f_{n+\frac{2}{3}}+\frac{43}{896}h^2f_{n+\frac{3}{4}}+\frac{10441}{6881280}h^2f_{n+1}\\
&y_{n+\frac{1}{3}}-\frac{4}{3}y_{n+\frac{1}{2}}+\frac{1}{3}y_{n+1}=\frac{22819}{44089920}h^2f_n-\frac{20}{1701}h^2f_{n+\frac{1}{6}}+\frac{25052}{688905}h^2f_{n+\frac{1}{4}}-\frac{6737}{181440}h^2f_{n+\frac{1}{3}}+\frac{12182}{229635}h^2f_{n+\frac{1}{2}}-\frac{10019}{544320}h^2f_{n+\frac{2}{3}}+\frac{628}{19683}h^2f_{n+\frac{3}{4}}+\frac{44659}{44089920}h^2f_{n+1}\\
&y_{n+\frac{2}{3}}-\frac{2}{3}y_{n+\frac{1}{2}}-\frac{1}{3}y_{n+1}=-\frac{45149}{88179840}h^2f_n+\frac{20}{1701}h^2f_{n+\frac{1}{6}}-\frac{5098}{137781}h^2f_{n+\frac{1}{4}}+\frac{2963}{72576}h^2f_{n+\frac{1}{3}}-\frac{7211}{229635}h^2f_{n+\frac{1}{2}}+\frac{24061}{1088640}h^2f_{n+\frac{2}{3}}-\frac{22418}{688905}h^2f_{n+\frac{3}{4}}-\frac{88829}{88179840}f_{n+1}\\
&y_{n+\frac{3}{4}}-\frac{1}{2}y_{n+\frac{1}{2}}-\frac{1}{2}y_{n+1}=-\frac{5179}{6881280}h^2f_n+\frac{4941}{286720}h^2f_{n+\frac{1}{6}}-\frac{727}{13440}h^2f_{n+\frac{1}{4}}+\frac{136323}{2293760}h^2f_{n+\frac{1}{3}}-\frac{37883}{860160}h^2f_{n+\frac{1}{2}}+\frac{18873}{458752}h^2f_{n+\frac{2}{3}}-\frac{109}{2240}h^2f_{n+\frac{3}{4}}-\frac{10373}{6881280}f_{n+1}
\end{aligned}\right\}\quad(8)$$

differentiating (6) once, yields

$$y'(t)=\sigma_{\frac{1}{2}}(t)y_{n+\frac{1}{2}}+\sigma_1(t)y_{n+1}+h^2\left[\sum_{j=0}^{1}\beta'_j(t)f_{n+j}+\beta'_{v_i}(t)f_{n+v_i}\right],v_i=0,\frac{1}{6},\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{3}{4}\qquad(9)$$

the coefficient $\sigma_{\frac{1}{2}},\sigma_1,\beta'_0,\beta'_{\frac{1}{6}},\beta'_{\frac{1}{4}},\beta'_{\frac{1}{3}},\ \beta'_{\frac{1}{2}},\beta'_{\frac{2}{3}},\beta'_{\frac{3}{4}},\beta'_1$ are given by;

$$\left.\begin{aligned}
&\alpha'_{\frac{1}{2}}=-2\\
&\alpha'_1=2\\
&\beta'_0=-\frac{1867}{40320}+t-\frac{113}{12}t^2+\frac{427}{9}t^3-\frac{3367}{24}t^4+\frac{3752}{15}t^5-\frac{791}{3}t^6+\frac{1056}{7}t^7-36t^8\\
&\beta'_{\frac{1}{6}}=-\frac{54}{175}+\frac{1944}{35}t^2-\frac{2376}{5}t^3+\frac{9072}{5}t^4-\frac{18792}{5}t^5+\frac{21816}{5}t^6-\frac{93312}{35}t^7+\frac{23328}{35}t^8\\
&\beta'_{\frac{1}{4}}=\frac{26}{175}-\frac{512}{5}t^2+\frac{45568}{45}t^3-\frac{21248}{5}t^4+\frac{28160}{3}t^5-\frac{171008}{15}t^6+\frac{251904}{35}t^7-\frac{9216}{5}t^8\\
&\beta'_{\frac{1}{3}}=-\frac{567}{3200}+\frac{729}{10}t^2-\frac{1539}{2}t^3+\frac{138267}{40}t^4-\frac{40338}{5}t^5+\frac{51111}{5}t^6-\frac{46656}{7}t^7+\frac{8748}{5}t^8\\
&\beta'_{\frac{1}{2}}=-\frac{143}{22400}+\frac{1}{10}t^2-\frac{107}{90}t^3+\frac{249}{40}t^4-\frac{262}{15}t^5+\frac{407}{15}t^6-\frac{768}{35}t^7+\frac{36}{5}t^8\\
&\beta'_{\frac{2}{3}}=-\frac{1647}{22400}+\frac{243}{20}t^2-\frac{702}{5}t^3+\frac{28269}{40}t^4-\frac{9396}{5}t^5+\frac{13581}{5}t^6-\frac{69984}{35}t^7+\frac{2916}{5}t^8\\
&\beta'_{\frac{3}{4}}=-\frac{278}{1575}-\frac{512}{105}t^2+\frac{512}{9}t^3-\frac{4352}{15}t^4+\frac{11776}{15}t^5-\frac{17408}{15}t^6+\frac{6144}{7}t^7-\frac{9216}{35}t^8\\
&\beta'_1=-\frac{143}{22400}+\frac{1}{10}t^2-\frac{107}{90}t^3+\frac{249}{40}t^4-\frac{626}{15}t^5+\frac{407}{15}t^6-\frac{768}{35}t^7+\frac{36}{5}t^8
\end{aligned}\right\}\quad(10)$$

on evaluating (9) at all point, so that the following discrete methods are obtained

$$\left.\begin{aligned}
&hy'_n+2y_{n+\frac{1}{2}}-2y_{n+1}=-\frac{1867}{40320}hf_n-\frac{54}{175}hf_{n+\frac{1}{6}}+\frac{26}{175}hf_{n+\frac{1}{4}}-\frac{567}{3200}hf_{n+\frac{1}{3}}-\frac{9}{35}hf_{n+\frac{1}{2}}+\frac{1647}{22400}hf_{n+\frac{2}{3}}-\frac{278}{1575}hf_{n+\frac{3}{4}}-\frac{143}{22400}hf_{n+1}\\
&hy'_{n+\frac{1}{6}}+2y_{n+\frac{1}{2}}-2y_{n+1}=-\frac{13609}{4898880}hf_n+\frac{167}{4725}hf_{n+\frac{1}{6}}-\frac{114598}{382725}hf_{n+\frac{1}{4}}+\frac{5611}{50400}hf_{n+\frac{1}{3}}-\frac{8789}{25515}hf_{n+\frac{1}{2}}-\frac{35107}{302400}hf_{n+\frac{2}{3}}-\frac{74014}{382725}hf_{n+\frac{3}{4}}-\frac{18511}{3061800}hf_{n+1}\\
&hy'_{n+\frac{1}{4}}+2y_{n+\frac{1}{2}}-2y_{n+1}=-\frac{1711}{573440}hf_n+\frac{24273}{358400}hf_{n+\frac{1}{6}}-\frac{23659}{100800}hf_{n+\frac{1}{4}}+\frac{273213}{2867200}hf_{n+\frac{1}{3}}-\frac{73343}{215040}hf_{n+\frac{1}{2}}+\frac{46953}{409600}hf_{n+\frac{2}{3}}-\frac{6479}{33600}hf_{n+\frac{3}{4}}-\frac{156281}{25804800}hf_{n+1}\\
&hy'_{n+\frac{1}{3}}+2y_{n+\frac{1}{2}}-2y_{n+1}=-\frac{811}{279936}hf_n+\frac{302}{4725}hf_{n+\frac{1}{6}}-\frac{72418}{382725}hf_{n+\frac{1}{4}}+\frac{28039}{201600}hf_{n+\frac{1}{3}}-\frac{8777}{25515}hf_{n+\frac{1}{2}}+\frac{69989}{604800}hf_{n+\frac{2}{3}}-\frac{73954}{382725}hf_{n+\frac{3}{4}}-\frac{296341}{48988800}hf_{n+1}\\
&hy'_{n+\frac{1}{2}}+2y_{n+\frac{1}{2}}-2y_{n+1}=-\frac{67}{20160}hf_n+\frac{27}{350}hf_{n+\frac{1}{6}}-\frac{386}{1575}hf_{n+\frac{1}{4}}+\frac{1539}{5600}hf_{n+\frac{1}{3}}-\frac{11}{42}hf_{n+\frac{1}{2}}+\frac{1161}{11200}hf_{n+\frac{2}{3}}-\frac{298}{1575}hf_{n+\frac{3}{4}}-\frac{11}{1800}hf_{n+1}\\
&hy'_{n+\frac{2}{3}}+2y_{n+\frac{1}{2}}-2y_{n+1}=-\frac{27569}{9797760}hf_n+\frac{302}{4725}hf_{n+\frac{1}{6}}-\frac{10894}{54675}hf_{n+\frac{1}{4}}+\frac{436339}{201600}hf_{n+\frac{1}{3}}-\frac{3713}{25515}hf_{n+\frac{1}{2}}+\frac{116789}{604800}hf_{n+\frac{2}{3}}-\frac{77794}{382725}hf_{n+\frac{3}{4}}-\frac{292261}{48988800}hf_{n+1}\\
&hy'_{n+\frac{3}{4}}+2y_{n+\frac{1}{2}}-2y_{n+1}=-\frac{3067}{1032192}hf_n+\frac{24273}{358400}hf_{n+\frac{1}{6}}-\frac{2371}{11200}hf_{n+\frac{1}{4}}+\frac{662013}{2867200}hf_{n+\frac{1}{3}}-\frac{11349}{71680}hf_{n+\frac{1}{2}}+\frac{717471}{2867200}hf_{n+\frac{2}{3}}-\frac{17117}{100800}hf_{n+\frac{3}{4}}-\frac{17329}{2867200}hf_{n+1}\\
&hy'_{n+1}+2y_{n+\frac{1}{2}}-2y_{n+1}=-\frac{61}{4480}hf_n-\frac{54}{175}hf_{n+\frac{1}{6}}+\frac{1514}{1575}hf_{n+\frac{1}{4}}-\frac{23409}{22400}hf_{n+\frac{1}{3}}+\frac{11}{15}hf_{n+\frac{1}{2}}-\frac{17793}{22400}hf_{n+\frac{2}{3}}+\frac{334}{525}hf_{n+\frac{3}{4}}+\frac{10793}{201600}hf_{n+1}
\end{aligned}\right\}\quad(11)$$

now, combining (8) and (11) in the block form to yield the block hybrid method, which can be written as

$$\left.\begin{aligned}
y_{n+\frac{1}{6}} &= y_n+\frac{1}{6}hy'_n+\frac{1000061}{176359680}h^2f_n+\frac{1247}{42525}h^2f_{n+\frac{1}{6}}-\frac{150733}{3444525}h^2f_{n+\frac{1}{4}}+\frac{104833}{3628800}h^2f_{n+\frac{1}{3}}-\frac{409}{45927}h^2f_{n+\frac{1}{2}}+\frac{47623}{10886400}h^2f_{n+\frac{2}{3}}-\frac{5989}{3444525}h^2f_{n+\frac{3}{4}}+\frac{30853}{881798400}h^2f_{n+1}\\
y_{n+\frac{1}{4}} &= y_n+\frac{1}{4}hy'_n+\frac{191741}{20643840}h^2f_n+\frac{85887}{1433600}h^2f_{n+\frac{1}{6}}-\frac{379}{4800}h^2f_{n+\frac{1}{4}}+\frac{599157}{11468800}h^2f_{n+\frac{1}{3}}-\frac{13789}{860160}h^2f_{n+\frac{1}{2}}+\frac{90099}{11468800}h^2f_{n+\frac{2}{3}}-\frac{629}{201600}h^2f_{n+\frac{3}{4}}+\frac{719}{11468800}h^2f_{n+1}\\
y_{n+\frac{1}{3}} &= y_n+\frac{1}{3}hy'_n+\frac{14221}{1102248}h^2f_n+\frac{3874}{42525}h^2f_{n+\frac{1}{6}}-\frac{373376}{3444525}h^2f_{n+\frac{1}{4}}+\frac{617}{8100}h^2f_{n+\frac{1}{3}}+\frac{5314}{229635}h^2f_{n+\frac{1}{2}}+\frac{3853}{340200}h^2f_{n+\frac{2}{3}}-\frac{15488}{3444525}h^2f_{n+\frac{3}{4}}+\frac{311}{3444525}h^2f_{n+1}\\
y_{n+\frac{1}{2}} &= y_n+\frac{1}{2}hy'_n+\frac{1621}{80640}h^2f_n+\frac{27}{175}h^2f_{n+\frac{1}{6}}-\frac{89}{525}h^2f_{n+\frac{1}{4}}+\frac{6399}{44800}h^2f_{n+\frac{1}{3}}-\frac{1}{30}h^2f_{n+\frac{1}{2}}+\frac{783}{44800}h^2f_{n+\frac{2}{3}}-\frac{11}{1575}h^2f_{n+\frac{3}{4}}+\frac{19}{134400}h^2f_{n+1}\\
y_{n+\frac{2}{3}} &= y_n+\frac{2}{3}hy'_n+\frac{18812}{688905}h^2f_n+\frac{9248}{42525}h^2f_{n+\frac{1}{6}}-\frac{796672}{3444525}h^2f_{n+\frac{1}{4}}+\frac{3022}{14175}h^2f_{n+\frac{1}{3}}-\frac{5024}{229635}h^2f_{n+\frac{1}{2}}+\frac{166}{6075}h^2f_{n+\frac{2}{3}}-\frac{34816}{3444525}h^2f_{n+\frac{3}{4}}+\frac{682}{3444525}h^2f_{n+1}\\
y_{n+\frac{3}{4}} &= y_n+\frac{3}{4}hy'_n+\frac{14187}{458752}h^2f_n+\frac{356481}{1433600}h^2f_{n+\frac{1}{6}}-\frac{5841}{22400}h^2f_{n+\frac{1}{4}}+\frac{2827791}{11468800}h^2f_{n+\frac{1}{3}}-\frac{3753}{286720}h^2f_{n+\frac{1}{2}}+\frac{461457}{11468800}h^2f_{n+\frac{2}{3}}-\frac{129}{11200}h^2f_{n+\frac{3}{4}}+\frac{2637}{11468800}h^2f_{n+1}\\
y_{n+1} &= y_n+hy'_n+\frac{109}{2520}h^2f_n+\frac{54}{175}h^2f_{n+\frac{1}{6}}-\frac{128}{525}h^2f_{n+\frac{1}{4}}+\frac{81}{350}h^2f_{n+\frac{1}{3}}-\frac{2}{21}h^2f_{n+\frac{1}{2}}+\frac{27}{1400}h^2f_{n+\frac{2}{3}}-\frac{128}{1575}h^2f_{n+\frac{3}{4}}+\frac{1}{300}h^2f_{n+1}\\
y'_{n+\frac{1}{6}} &= y'_n+\frac{426463}{9797760}hf_n+\frac{65}{189}hf_{n+\frac{1}{6}}-\frac{34292}{76545}hf_{n+\frac{1}{4}}+\frac{11633}{40320}hf_{n+\frac{1}{3}}-\frac{2228}{25515}hf_{n+\frac{1}{2}}+\frac{5149}{120960}hf_{n+\frac{2}{3}}-\frac{1292}{76545}hf_{n+\frac{3}{4}}+\frac{3313}{9797760}hf_{n+1}\\
y'_{n+\frac{1}{4}} &= y'_n+\frac{223577}{5160960}hf_n+\frac{26973}{71680}hf_{n+\frac{1}{6}}-\frac{7727}{20160}hf_{n+\frac{1}{4}}+\frac{156249}{573440}hf_{n+\frac{1}{3}}-\frac{18047}{215040}hf_{n+\frac{1}{2}}+\frac{23571}{573440}hf_{n+\frac{2}{3}}-\frac{47}{2880}hf_{n+\frac{3}{4}}+\frac{1691}{5160960}hf_{n+1}\\
y'_{n+\frac{1}{3}} &= y'_n+\frac{26581}{612360}hf_n+\frac{352}{945}hf_{n+\frac{1}{6}}-\frac{25856}{76545}hf_{n+\frac{1}{4}}+\frac{797}{2520}hf_{n+\frac{1}{3}}-\frac{2216}{25515}hf_{n+\frac{1}{2}}+\frac{319}{7560}hf_{n+\frac{2}{3}}-\frac{256}{15309}hf_{n+\frac{3}{4}}+\frac{41}{122472}hf_{n+1}\\
y'_{n+\frac{1}{2}} &= y'_n+\frac{1733}{40320}hf_n+\frac{27}{70}hf_{n+\frac{1}{6}}-\frac{124}{315}hf_{n+\frac{1}{4}}+\frac{405}{896}hf_{n+\frac{1}{3}}-\frac{1}{210}hf_{n+\frac{1}{2}}+\frac{27}{896}hf_{n+\frac{2}{3}}-\frac{4}{315}hf_{n+\frac{3}{4}}+\frac{11}{40320}hf_{n+1}\\
y'_{n+\frac{2}{3}} &= y'_n+\frac{3329}{76545}hf_n+\frac{352}{945}hf_{n+\frac{1}{6}}-\frac{26624}{76545}hf_{n+\frac{1}{4}}+\frac{124}{315}hf_{n+\frac{1}{3}}-\frac{2848}{25515}hf_{n+\frac{1}{2}}+\frac{113}{945}hf_{n+\frac{2}{3}}-\frac{2048}{76545}hf_{n+\frac{3}{4}}+\frac{32}{76545}hf_{n+1}\\
y'_{n+\frac{3}{4}} &= y'_n+\frac{24849}{573440}hf_n+\frac{26973}{71680}hf_{n+\frac{1}{6}}-\frac{807}{2240}hf_{n+\frac{1}{4}}+\frac{234009}{573440}hf_{n+\frac{1}{3}}-\frac{7083}{71680}hf_{n+\frac{1}{2}}+\frac{101331}{573440}hf_{n+\frac{2}{3}}-\frac{3}{448}hf_{n+\frac{3}{4}}+\frac{39}{114688}hf_{n+1}\\
y'_{n+1} &= y'_n+\frac{151}{2520}hf_n+\frac{256}{315}hf_{n+\frac{1}{4}}-\frac{243}{280}hf_{n+\frac{1}{3}}+\frac{104}{105}hf_{n+\frac{1}{2}}-\frac{243}{280}hf_{n+\frac{2}{3}}+\frac{256}{315}hf_{n+\frac{3}{4}}+\frac{151}{2520}hf_{n+1}
\end{aligned}\right\}\qquad (12)$$

3. Results Analysis of Computational Method

In this section, the analysis of the basic properties of the new method are analyzed. These properties are order, error constant, consistency, zero-stability and region of absolute stability.

3.1. Order and Error Constant of the computational method

Let the linear operator defined on the method be $\ell\left[y(t);h\right]$, where,

$$\Delta\{y(t):h\}=A^{(0)}Y_m^{(i)}-\sum_{i=0}^{k}\frac{jh^{(i)}}{i!}y_n^{(i)}-h^{(3-1)}\left[d_i f(y_n)+b_iF(Y_m)\right] \qquad (13)$$

Expanding $Y_m\ and\ F(Y_m)$ in Taylor series and comparing the coefficients of h gives

$$\Delta\{y(t):h\}=C_0y(t)+C_1y'(t)+\cdots+C_ph^py^p(t)+C_{p+1}h^{p+1}y^{p+1}(t)+C_{p+2}h^{p+2}y^{p+2}(t)+\cdots \qquad (14)$$

Definition 1: The linear operator L and the associate block method are said to be of order p if $C_0=C_1=\cdots=C_p=C_{p+1}=0,\quad C_{p+2}\neq 0.\quad C_{p+2}$ is called the error constant and implies that the truncation error is given by $t_{n+k}=C_{p+2}h^{p+2}y^{p+3}(t)+0h^{p+3}$

$$L\{y(t):h\}=C_0y(t)+C_1y'(t)+\cdots+C_ph^py^p(t)+C_{p+1}h^{p+1}y^{p+1}(t)+C_{p+2}h^{p+2}y^{p+2}(t)+\cdots \qquad (15)$$

$$\left[\begin{matrix}
y\sum_{j=0}^{\infty}\frac{\left(\frac{1}{6}\right)^{j}}{j!}-y_n-\frac{1}{6}hy'_n-\frac{1000061}{176359680}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\frac{1247}{42525}\left(\frac{1}{6}\right)-\frac{150733}{3444525}\left(\frac{1}{4}\right)+\frac{104833}{3628800}\left(\frac{1}{3}\right)-\frac{409}{45927}\left(\frac{1}{2}\right)+\frac{47623}{10886400}\left(\frac{2}{3}\right)-\frac{5989}{3444525}\left(\frac{3}{4}\right)+\frac{30853}{881798400}(1)\right]\\
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{4}\right)^{j}}{j!}-y_n-\frac{1}{4}hy'_n-\frac{191741}{20643840}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\frac{85887}{1433600}\left(\frac{1}{6}\right)-\frac{379}{4800}\left(\frac{1}{4}\right)+\frac{599157}{11468800}\left(\frac{1}{3}\right)-\frac{13789}{860160}\left(\frac{1}{2}\right)+\frac{90099}{11468800}\left(\frac{2}{3}\right)-\frac{629}{201600}\left(\frac{3}{4}\right)+\frac{719}{11468800}(1)\right]\\
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{3}\right)^{j}}{j!}-y_n-\frac{1}{3}hy'_n-\frac{14221}{1102248}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\frac{3874}{42525}\left(\frac{1}{6}\right)-\frac{373376}{3444525}\left(\frac{1}{4}\right)+\frac{617}{8100}\left(\frac{1}{3}\right)=\frac{5314}{229635}\left(\frac{1}{2}\right)+\frac{3853}{340200}\left(\frac{2}{3}\right)-\frac{15488}{3444525}\left(\frac{3}{4}\right)+\frac{311}{3444525}(1)\right]\\
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{j}}{j!}-y_n-\frac{1}{2}hy'_n-\frac{1621}{80640}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\frac{27}{175}\left(\frac{1}{6}\right)-\frac{89}{525}\left(\frac{1}{4}\right)+\frac{6399}{44800}\left(\frac{1}{3}\right)-\frac{1}{30}\left(\frac{1}{2}\right)+\frac{783}{44800}\left(\frac{2}{3}\right)-\frac{11}{1575}\left(\frac{3}{4}\right)+\frac{19}{134400}h^{2}f_{n+1}\right]\\
\sum_{j=0}^{\infty}\frac{\left(\frac{2}{3}\right)^{j}}{j!}-y_n-\frac{2}{3}hy'_n-\frac{18812}{688905}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\frac{9248}{42525}\left(\frac{1}{6}\right)-\frac{796672}{3444525}\left(\frac{1}{4}\right)+\frac{3022}{14175}\left(\frac{1}{3}\right)-\frac{5024}{229635}\left(\frac{1}{2}\right)+\frac{166}{6075}\left(\frac{2}{3}\right)-\frac{34816}{3444525}\left(\frac{3}{4}\right)+\frac{682}{3444525}(1)\right]\\
\sum_{j=0}^{\infty}\frac{\left(\frac{3}{4}\right)^{j}}{j!}-y_n-\frac{3}{4}hy'_n-\frac{14187}{458752}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\frac{356481}{1433600}\left(\frac{1}{6}\right)-\frac{5841}{22400}\left(\frac{1}{4}\right)+\frac{2827791}{11468800}\left(\frac{1}{3}\right)-\frac{3753}{286720}\left(\frac{1}{2}\right)+\frac{461457}{11468800}\left(\frac{2}{3}\right)-\frac{129}{11200}\left(\frac{3}{4}\right)+\frac{2637}{11468800}(1)\right]\\
\sum_{j=0}^{\infty}\frac{(1)^{j}}{j!}-y_n-hy'_n-\frac{109}{2520}hy''_n-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\frac{54}{175}\left(\frac{1}{6}\right)-\frac{128}{525}\left(\frac{1}{4}\right)+\frac{81}{350}\left(\frac{1}{3}\right)-\frac{2}{21}\left(\frac{1}{2}\right)+\frac{27}{1400}\left(\frac{2}{3}\right)-\frac{128}{1575}\left(\frac{3}{4}\right)+\frac{1}{300}(1)\right]
\end{matrix}\right]=0$$

Comparing the coefficient of h, according to [7], the new method is of uniform order $p=\begin{bmatrix}7 & 7 & 7 & 7 & 7 & 7 & 7\end{bmatrix}^{T}$ with its error constant are given respectively by

$$C_{p+2}=\begin{bmatrix}-1.3640\times10^{-10} & -1.3281\times10^{-10} & -1.3497\times10^{-10} & -1.1961\times10^{-10} & -1.4526\times10^{-10} & -1.3561\times10^{-10} & -1.4353\times10^{-09}\end{bmatrix}$$

3.2. Consistency of the Method

A numerical method is said to be consistent if the following conditions are satisfied.

i. The order of the method must be greater than or equal to zero to one i.e. $p \geq 1$.

ii. $\sum_{j=0}^{k}\alpha_j = 0$

iii. $\rho(r)=\rho'(r)=0$

iv. $\rho'''(r)=3!\sigma(r)$

Where $\rho(r)$ *and* $\sigma(r)$ are first and second characteristics polynomials of our method. According to [7], the first condition is a sufficient condition for the associated block method to be consistent. Hence the new method is consistent.

3.3. Zero Stability of the Method

Definition 2: the numerical method is said to be zero-stable, if the roots $z_s,\ s=1,2,\cdots,k$ of the first characteristics polynomial $\rho(z)$ defined by $\rho(z)=\det\left(zA^{(0)}-E\right)$ satisfies $|z_s|\leq 1$ and every root satisfies $|z_s|=1$ have multiplicity not exceeding the order of the differential equation [19]. The first characteristic polynomial is given by,

$$\rho(z)=\left| z\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\end{bmatrix}\right|=\left\|\begin{bmatrix}z&0&0&0&0&0&-1\\0&z&0&0&0&0&-1\\0&0&z&0&0&0&-1\\0&0&0&z&0&0&-1\\0&0&0&0&z&0&-1\\0&0&0&0&0&z&-1\\0&0&0&0&0&0&z-1\end{bmatrix}\right\|=z^6(z-1)$$

Thus, solving for z in

$$z^6(z-1) \qquad (16)$$

gives $z=0,0,0,0,0,0,1$. Hence the new method is said to be zero-stable.

3.4. Convergence of the Block Method

Theorem 1: the necessary and sufficient conditions for linear multistep method to be convergent are that it must be consistent and zero-stable. Hence the new method formulated is consistent [7].

3.5. Region of Absolute Stability of our Method

Definition 3: the region of absolute stability is the region of the complex z plane, where $z=\lambda h$ for which the method is absolute stable. To determine the region of absolute stability of the block method, the methods that compare neither the computation of roots of a polynomial nor solving of simultaneous inequalities was adopted. Thus, the method according to [19] is called the boundary locus method. Applying this method we obtain the stability polynomial as

$$\bar{h}(w)=\left(-\frac{1}{11612160}w^6-\frac{163}{14631321600}w^7\right)h^{14}+\left(-\frac{810757}{87787929600}w^6+\frac{113}{34836480}w^7\right)h^{12}+\left(-\frac{4473503}{37623398400}w^6-\frac{61}{829440}w^7\right)h^{10} \qquad (17)$$
$$+\left(-\frac{1235057}{522547200}w^6+\frac{481}{414720}w^7\right)h^8+\left(-\frac{128809}{6967296}w^6-\frac{67}{5184}w^7\right)h^6+\left(-\frac{817}{5040}w^6+\frac{113}{1152}w^7\right)h^4+\left(-\frac{17}{30}w^6-\frac{11}{24}w^7\right)h^2-2w^6+w^7$$

Using the stability polynomial (17), we obtain the region of absolute stability in figure below as

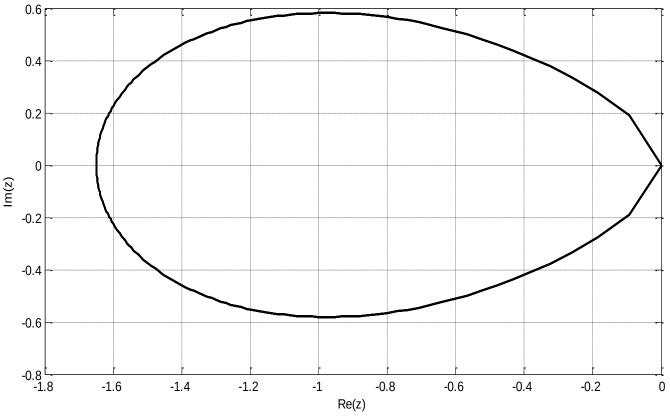

Figure 1. Stability Region.

The stability region obtained in Figure 1 is $A_\alpha-stable$.

4. Numerical Applications of the Computational Method

The section tests the computational method through its application to second order oscillatory differential equations which take the form of (1). The results of the comparison between the new computational method and the existing method from [6, 20] are displayed through tabulated data and graphical representations. The following notation shall be used in the tables and figures.

t means the points of evaluation

ES means Exact Solution

CS means Computed solution

AECM means Absolute Error in Computational method

AE[5] means Absolute Error in [5]

AE[20] means Absolute Error in [20]

Example 1: Consider the Second Order Oscillatory Betiss Differential Equation of the form

$$\frac{d^2y}{dt^2}+\frac{dy}{dt}=0.001\cos(t),\ y(0)=1,\ \frac{dy}{dt}=0 \qquad (18)$$

With exact solution as

$$y(t)=\cos(t)+0.0005\,t\sin(t) \qquad (19)$$

Source [5, 20].

Table 1. Showing the Numerical Results

t	**ES**	**CS**	**AECM**	**AE[5]**	**AE[20]**
0.1	0.09978366643856425102	0.09978366643856425102	0.0000E00	1.2567E-12	1.0170E-12
0.2	0.19857132413727709130	0.19857132413727709132	2.0000E-20	2.1140E-12	1.4285E-11
0.3	0.29537690618797073421	0.29537690618797073422	1.0000E-20	2.3764E-12	4.9557E-11
0.4	0.38923413010984991465	0.38923413010984991467	2.0000E-20	3.4242E-12	1.0161E-10
0.5	0.47920614296373040709	0.47920614296373040711	2.0000E-20	3.3944E-12	1.7416E-10
0.6	0.56439487271056245371	0.56439487271056245372	1.0000E-20	3.3436E-12	2.6425E-10
0.7	0.64394999247214148272	0.64394999247214148273	1.0000E-20	4.2949E-12	3.7579E-10
0.8	0.71707740821578389546	0.71707740821578389546	0.0000E00	4.2574E-12	5.0602E-10
0.9	0.78304718514176158945	0.78304718514176158945	0.0000E00	5.2344E-12	6.5904E-10
1.0	0.84120083365496243679	0.84120083365496243678	1.0000E-20	6.2265E-12	8.3225E-10

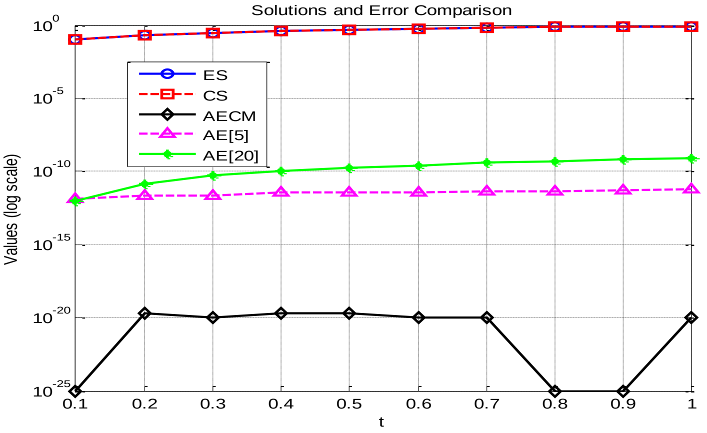

Figure 2. Textual Curve of table 1

Example 2: Consider the Second Order Oscillatory Stiefel Differential Equation

$$\frac{d^2y}{dt^2}+\frac{dy}{dt}=0.001\sin(t),\ y(0)=0,\ \frac{dy}{dt}=0.9995 \qquad (20)$$

With exact solution of as

$$y(t)=\sin(t)-0.0005\,t\cos(t) \qquad (21)$$

Source [5, 20].

Table 2. Showing the Numerical Results for Example 2

	ES	**CS**	**AECM**	**AE[5]**	**AE[20]**
0.1	0.99500915694885810751	0.99500915694885810750	1.0000E-20	2.8269E-12	1.0169E-11
0.2	0.98008644477432113724	0.98008644477432113723	1.0000E-20	5.8994E-12	2.0390E-11
0.3	0.95538081715660522058	0.95538081715660522057	1.0000E-20	6.8309E-12	1.5451E-13
0.4	0.92113887767134681290	0.92113887767134681288	2.0000E-20	1.4991E-12	8.1063E-11
0.5	0.87770241827502376687	0.87770241827502376685	2.0000E-20	1.8395E-12	2.5377E-10
0.6	0.82550500765169680785	0.82550500765169680783	2.0000E-20	1.6559E-11	5.4848E-10
0.7	0.76506766347502161813	0.76506766347502161811	2.0000E-20	1.2970E-11	9.9571E-10
0.8	0.69699365178352523002	0.69699365178352523001	1.0000E-20	8.4312E-11	1.6260E-10
0.9	0.62196246537999682400	0.62196246537999682400	0.0000E00	5.3240E-11	2.4697E-10
1.0	0.54072304136054366565	0.54072304136054366565	0.0000E00	3.2126E-11	3.5575E-10

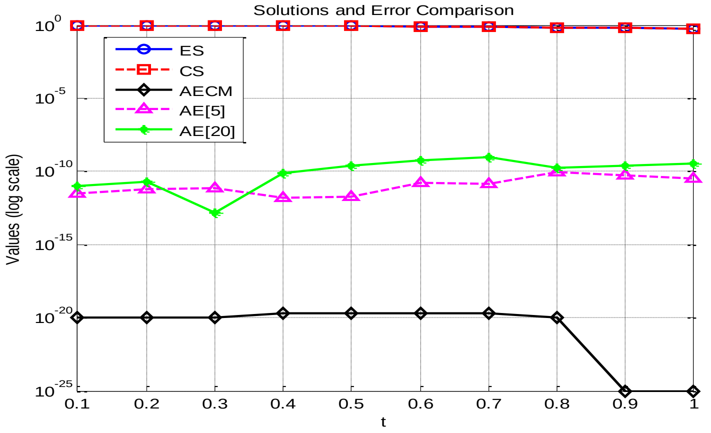

Figure 3. Textual Curve of Figure 1

5. Discussion and Results

The mathematical formulation of the computational method begins by approximating the solution of a second-order initial value problem using a power series polynomial as a basis function. The polynomial contains unknown real parameters which researchers define by selecting specific interpolation and collocation points. The process begins with the polynomial's second derivative because it creates a set of linear equations which engineers represent in matrix format. The solution process involves using Gaussian elimination to evaluate unknown parameters which researchers insert into the original problem to create a continuous hybrid linear multistep method. The method assessment at both interpolating and non-interpolating points results in a discrete representation which combines with continuous forms to create a block hybrid method that enables efficient numerical calculations.

The study investigates the basic numerical characteristics of the computational method through its mathematical assessment. The method's order and error constant are determined through applying a linear operator which requires Taylor series expansion to achieve minimized truncation errors. The method's consistency verification depends on two conditions which involve its method order and the behavior of its characteristic polynomials. The analysis of the first characteristic polynomial roots shows that the method achieves zero-stability because all roots meet the required conditions. The linear multistep methods theorem guarantees that numerical solutions for second-order initial value problems will converge because the method achieves both consistency and zero-stability.

The boundary locus method establishes the absolute stability region of the block method without requiring any complex root computations or inequality resolution procedures. The stability polynomial derived through this process defines the absolute stable region of the method in the complex plane. The stability region shown in Figure 1 demonstrates that the method maintains its effectiveness across different step sizes and problem parameters. The block hybrid method which was developed demonstrates reliability and computational effectiveness as a numerical solution tool for second-order differential equations because it combines high-order performance with consistency and zero-stability properties and a clearly established stability region.

The results presented in Table 4.1 for Example 1 demonstrate that the computational method AECM successfully solves the Second Order Oscillatory Betiss Differential Equation. The table presents a comparison between the exact solution (ES) and the computed solution (CS) which was derived through AECM and two other methods from reference [5, 20]. The computed solutions match the exact solutions at all independent variable values which demonstrates that the computational method successfully models system behavior.

The graphical representation found in Figure 2 serves as visual proof of the numerical findings. The exact solution and the computed solution through AECM show almost complete overlap which demonstrates their results match with high precision. The visual alignment demonstrates the proposed method's reliability because it successfully models oscillatory behavior and produces stable outcomes during real-world testing.

The results of Table 2 demonstrate that the AECM method for the Second Order Oscillatory Stiefel Differential Equation produces solutions which closely match the exact solution. The computed solution almost matches the exact solution according to the textual curve in Figure 2 which shows this agreement. The combination of tables and figures proves that the proposed computational method demonstrates strong performance through its accurate results which enable it to handle multiple second-order oscillatory differential equations.

░ 6. Summary and Conclusion

The researchers developed an effective computational solution method which directly solves second-order oscillatory differential equations through their study of the Betiss and Stiefel mathematical forms. The approach uses power series polynomial functions as its basis to create a continuous linear multistep hybrid system through interpolation and collocation methods. The method underwent complete analysis to determine its numerical characteristics which included order analysis, error constant assessment, consistency evaluation, zero-stability measurement, convergence testing, and absolute stability region determination. The numerical experiments proved that the proposed method achieves results which strongly match exact solutions while performing better than some current numerical methods. The method demonstrated accurate performance with efficient operation and stable results which researchers confirmed through both tabular and graphical comparisons across multiple test problems.

The computational method presented in this study is a reliable and efficient tool for solving second-order oscillatory differential equations. Its high-order accuracy, consistency, and zero-stability, combined with a clearly defined region of absolute stability, make it suitable for practical use in all three fields of mathematics, physics, and engineering. The method demonstrates effective performance because numerical results match the exact solutions with excellent accuracy. The computational method establishes a solid theoretical and practical foundation which enables accurate and efficient simulation of systems that follow complex oscillatory differential equations.

░ 7. Suggestions for Future Studies

Future Research can use:

1. Future work can extend the developed method to solve higher-order oscillatory differential equations (such as third- and fourth-order problems), thereby broadening its applicability in advanced mathematical and engineering models.
2. The method can be further developed to handle nonlinear and stiff oscillatory differential equations, which are common in real-life applications but more challenging to solve numerically.

3. Introducing an adaptive or variable step-size strategy into the block hybrid method would improve accuracy and computational efficiency, especially for problems with varying solution behavior.
4. Future research can focus on implementing the method in computational software such as MATLAB, Python, or Mathematica to enable large-scale simulations and practical usability.
5. A detailed comparison with other advanced numerical techniques, such as Runge-Kutta methods, spectral methods, and machine learning-based approaches, can be conducted to evaluate performance, efficiency, and accuracy.

Declarations

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Competing Interests Statement

The authors have not declared any conflict of interest.

Consent for publication

The authors declare that they consented to the publication of this study.

Authors' contributions

All the authors took part in literature review, analysis and manuscript writing equally.

Informed Consent

Not applicable for this study.

Availability of data and material

Supplementary information is available from the authors upon request.

Institutional Review Board Statement

Not applicable for this study.

References

[1] Sabo, J., Kyagya, T.Y., & Vashawa, W.J. (2021). Numerical simulation of one-step block method for treatment of second-order forced motions in mass-spring systems. Asian Journal of Research and Reviews in Physics, 5(2): 1–11.

[2] Elazzouzi, A., Lamrani, A.A., Tilioua, M., & Torres, D.F.M. (2019). Analysis of a SIRI epidemic model with distributed delay and relapse. Statistics, Optimization & Information Computing, 7: 545–557.

[3] Kuboye, J.O., & Omar, Z. (2015). Derivation of a six-step block method for direct solution of second-order ordinary differential equations. Mathematical and Computational Applications, 20: 151–159.

[4] Kyagya, T.Y., Raymond, D., & Sabo, J. (2021). Numerical application of ordinary differential equations using power series for solving higher-order initial value problems. FUW Trends in Science and Technology, 6(3): 868–876.

[5] Lydia, J.K., Ndam, J.N., & James, A.A. (2021). On the numerical approximations and simulations of damped and undamped Duffing oscillators. Science Forum (Journal of Pure and Applied Science), 21: 503–515.

[6] Adeyeye, O., & Omar, Z. (2017). 4-step 5-point hybrid block method for the direct solution of second-order initial value problems. Journal of Mathematics and Computer Science, 17: 527–534.

[7] Skwame, Y., Sabo, J., & Mathew, M. (2019). The treatment of second-order ordinary differential equations using equidistant one-step block hybrid. Asian Journal of Probability and Statistics, 5(3): 1–9.

[8] Kwari, L.J., Sunday, J., Ndam, J.N., Shokri, A., & Wang, Y. (2023). On the simulations of second-order oscillatory problems with applications to physical systems. Axioms, 12(9): 10. https://doi.org/10.3390/axioms 12030282.

[9] Kwanamu, J.A., Skwame, Y., & Sabo, J. (2021). Block hybrid method for solving higher-order ordinary differential equation using power series on implicit one-step second derivative. FUW Trends in Science and Technology, 6(2): 576–582.

[10] Donald, J.Z., Skwame, Y., Sabo, J., & Ayinde, A.M. (2021). The use of linear multistep method on implicit one-step second derivative block method for direct solution of higher-order initial value problems. Abacus (Mathematics Science Series), 48(2): 224–237.

[11] Ayinde, A.M., Ibrahim, S., Sabo, J., & Silas, D. (2023). The physical application of motion using single-step block method. Journal of Material Science Research and Review, 6(4): 708–719.

[12] Ishaq, A.A., Ayinde, M.M., Sabo, J., Ibrahim, S., & Jaiyeola, B.S. (2024). Three-step collocation method for solution of third derivative initial value problem. International Journal of Development Mathematics, 1(1): 25–36. https://doi.org/10.62054/ijdm/0101.03.

[13] Adewale, A.J., & Sabo, J. (2024). Simulating the dynamics of oscillating differential equations of mass in motion. International Journal of Development Mathematics, 1(1): 54–69. https://doi.org/10.62054/ijdm/0101.07.

[14] Aloko, M.D., Ayinde, M.A., Usman, A.D., & Sabo, J. (2024). An efficient scheme for direct simulation of third and fourth oscillatory differential equations. International Journal of Development Mathematics, 1(1): 72–89. https://doi.org/10.62054/ijdm/0101.06.

[15] Sabo, J., Ayinde, A.M., Oyedepo, T., & Ishaq, A.A. (2024). Simulation of two-step block approach for solving oscillatory differential equations. Cumhuriyet Science Journal. https://doi.org/10.17776/csj.1345303.

[16] Raymond, D., Skwame, Y., & Sunday, J. (2018). An implicit two-step one off-grid points third derivative hybrid block method for the direct solution of second-order ordinary differential equations. Academic Journal of Applied Mathematical Sciences, 4(2): 8–14.

[17] Ibrahim, I.H. (2017). Hybrid multistep method for solving second-order initial value problems. Journal of the Egyptian Mathematical Society, 25: 355–362.

[18] Kamo, N.M., Abada, A.A., & Soomiyol, M.C. (2018). A block procedure with continuous coefficients for direct solution of initial value problems of ODEs using shifted Legendre polynomial as basic function. International Journal of Multidisciplinary Research and Development, 5(4): 236–241.

[19] Sunday, J. (2018). On the oscillation criteria and computation of third-order oscillatory differential equations. Communication in Mathematics and Applications, 6(4): 615–625.

[20] Olabode, B.T., & Momoh, A.L. (2016). Continuous hybrid multistep methods with Legendre basic function for treatment of second-order stiff ODEs. American Journal of Computational and Applied Mathematics, 6(2): 38–49.